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A Study on an Extension of a Class of Generating Functions Involving the Biorthogonal Polynomials

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Abstract : In this paper we obtain a new class of quasi bilateral generating functions involving the Gegenbauer polynomial $C_s^n u$, Biorthogonal polynomial $T_{k-1,n}^{\alpha}(x)$ and Laguerre $L_m^n(Z)$ from the view point of the Lie-Algebra (i.e. Lie-group) and some known bilateral generating functions are also obtained as special cases.

Key words: Biorthogonal Polynomials, Gegenbauer Polynomials, Generating functions.

INTRODUCTION

Biorthogonal polynomial $T_{k-1,n}^{\alpha}(x)$ in terms of x^k by [3] in 1968, where k is a positive integer and a relation $y_n^{(k\alpha + k-1)}(x^k, k) = T_{k^{-1},n}^{\alpha}(x)$ where $y_n^{(\alpha)}(x\,;\,k)$ is a biorthogonal polynomial in x of degree n as defined by [2]. The following bilateral generating function of the biorthogonal polynomial $T_{k^{-1},n}^{\alpha}(x)$ obtained by [4] and [6]

$(1 - ty)^{-(k\alpha+k)} \exp\left[-tx^k y/(1-ty)\right] G\left\{x^k/(1-ty), ty/(1-ty)\right\}$

$$= \sum_{m=0}^{\infty} T_{k^{-1},m}^{\alpha}(x)\,\sigma_k(t)\,y^{nk+m-n} \qquad \ldots.(1.1)$$

Where $\sigma_k(t) = \sum_{n=0}^{m} a_n\, t^{nk}\,(nk+k)_{m-n}/(m-n)!$

The aim of this paper is to extend the bilateral generating function of the biorthogonal polynomial $T_{k^{-1},n}^{\alpha}(x)$ involving Laguerre polunomial $L_m^n(Z)$ and Gegenbauer polynomial $C_s^n(u)$. Our result can be put in the form of a theorem as follows:

Theorem-1

If the quasi bilateral generating function exists for the biorthogonal polynomial, Gegenbauer polynomial and Laguerre polynomial

$$G(x^k, u, z, w) = \sum_{n=0}^{\infty} a_n\, w^{nk}\, T^{\alpha}_{k^{-1},n}(x)\, C^n_x(u)\, L^n_m(Z)$$

then the following new general classes of generating functions are hold.

$$(1-wy)^{-(k\alpha+k)} \exp\,(-wk^k/(1-w)) \exp\{(-w)/(1-2w)^{-s/2}\}\times$$
$$G\{x^k/(1-w), u/\sqrt{1-2w}, z+w, wj/(1-2w)(1-wy)\}$$
$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} a_n \quad j^{nk}\, w^{p+\Sigma+r+nk} / p!\Sigma!r!(nk+k)_p \times$$
$$T^{\alpha}_{k^{-1};n+p}(x)\, 2^q\, (nk)_q\, C^{nk+q}_s(u)\, (-1)^r\, L^{nk+r}_m(z)$$

All the special classes of generating functions can be easily deduced by values for a_n and then making use of known quasi bilateral generating functions involving biorthogonal polynomial $T^{\alpha}_{k^{-1},n}(x)$

II. Proof of the Theorem

Let us suppose

$$G(x^k, u, z, w) = \sum_{n=0}^{\infty} a_n\, w^{nk}\, T^{\alpha}_{k^{-1},n}(x)\, C^n_x(u)\, L^n_m(z)$$

replace w by wytvj, then we get

$$G(x^k, u, z, wytvj)\, y^{nk} = \sum_{n=0}^{\infty} a_n\, (wj)^{nk}\, T^{\alpha}_{k^{-1},n}(x)\, y^{nk}\, C^n_s(u)\, v^{nk}\, L^n_m(z)\, t^{nk}$$

…..(2.1)

Now taking the following partial differential operators for the biorthogonal polynomial $T^{\alpha}_{k^{-1},n}(x)$ by [2]

$$R_1 = x^k y \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \left(k\alpha + k - x^k\right) y$$

such as

$$R_1 \left[T^{\alpha}_{k^{-1},n}(x)\, y^{nk}\right] = (kn+k)\, T^{\alpha}_{k^{-1},\,n+1}(x)\, y^{nk-1}$$

…..(2.2)

Moreover the extended form of the group generated by is given by

exp wR_1 f(x^k, y) = (1-wy)$^{-(k\alpha+k)}$ exp {-wx^ky/(1-wy)}. f{(x^k/(1-wy); y/(1-wy)}

……(2.3)

Again we taking the linear partial differential operator for the Gegennawer polynomial $C^n_s(u)$ by [5] as follows:

$$R_2 = uv \frac{\partial}{\partial u} + 2v^2 \frac{\partial}{\partial v} + sv$$

such as $R_2 \left[C^S_n(n)\, v^n\right] = 2n\, C^{n+1}_s(u)\, v^{n+1}$

……(2.4)

Similarly the extended form of the groups generated by R_2 is given by

exp w R_2 f(u, v) = (1-2 wv)$^{-s/2}$ f $\left(u/\sqrt{1-2wv},\, v/(1-2wv)\right)$

…...(2.5)

Again we taking the linear partial differential operator for the Laguerre polynomial $L^n_m(Z)$ by [1] as follows:

$$R_3 = t \frac{\partial}{\partial z} - t$$

Such as

$$R_3 \left[L^n_m(z)\, t^n\right] = -\, L^{n+1}_m(z)\, t^{n+1}$$

….. (2.6)

The extended form of the group generated by R_3 is given by

$$\exp wR_3\, f(z,t) = \exp(-wt)\, f(z+wt, t) \quad \text{....(2.7)}$$

Now multiply by exp wR_1 exp wR_2 exp wR_3 on both side of equation (2.1) then we get

$$\exp wR_1 \exp wR_2 \exp wR_3\, G(x^{k,}{}_{n, z, wy\, tv})$$

$$= \exp wR_1 \exp wR_2 \exp wR_3 \sum_{n=0}^{\infty} a_n\, (wj)^{nk}\, T^{\alpha}_{k^{-1},n}(x)\, y^{nk}\, C^n_s(u)\;\; v^{nk}\, L^n_m(z)\, t^{nk}$$

..... (2.8)

Now the L.H.S. of equation (2.8) with the help of (2.3), (2.5) and (2.7) we get

$$(1-wy)^{-(k\alpha+k)}\;\exp\{-wx^k y/(1-wy)\}\;\exp(-wt)\,(1-2wv)^{-s/2}$$
$$G\,\{x^k/(1-wy), u/\sqrt{1-2wv}, z+wt, wytvj/(1-2wv)\,(1-wy)\}$$

.....(2.9)

Again the R.H.S. of equation (2.8) with the help of equation (2.2), (2.4) and (2.6)

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} a_n\, f^{nk}\;\; w^{nk+p+q+r}/p!q!t!\,(nk+k)_p$$

$$\times\, T^{\alpha}_{k^{-1},n+p}(x)\, 2^q\, y^{nk+p(nk)_p}\, C^{nk+q}_m(u)\, v^{nk+q}\,(-1)^r\, L^{nk+r}_m(z)\, t^{nk+r}$$

.....(210)

equation the equation (2.9) and (2.10) we get equation (2.8) and substituting y = t = v = 1 we get equation (1.2)

III. Special case

It put m = s = 0 in (2.2) and using

$$C_0^{nk+q}(u) = L^{nk-r}_0(z) = 1$$

then we get

$$(1-w)^{-(k\alpha+k)} \exp(-wx^k/(1-w)) \exp(-w)\, G\,(x^k/(1-w), wj/(1-2w)(1-w))$$

$$= \sum_{r=0}^{\infty}(-w)^r/r!\sum_{q=0}^{\infty}(2w)^q/q!\,(nk)_q \sum_{n,\,p=0}^{\infty} a_n\; j^{nk} w^{p+nk}/\;p!\,(nk+k)_p\, T^{\alpha}_{k^{-1},n+p}(x)$$

Now replace j by j (1-2w). then we get

$$(1-w)^{-(k\alpha+k)} \exp\{(-wx^k/(1-w)\} G\left[x^k/(1-w), wj/1-w\right]$$

$$= \sum_{p=0}^{\infty} \sum_{n=0}^{p} a_n \, f^{nk} \, w^{p+nk-n} / (p-n)! \, (nk+k)_{p-n} \, T^{\alpha}_{k^{-1},\, p}(x)$$

….. (3.1)

Now put k = 1 in (3.1) and using

$$y_n^{k\alpha+k-1}(x^k, k) = T^{\alpha}_{k^{-1},\, n}(x) \text{ then we set}$$

$$(1-w)^{-(\alpha+1)} \exp\{-wx/(1-w)\} \, G\,(x/(1-w), wj/(1-w)]$$

$$= \sum_{p=0}^{\infty} L_p^{(\alpha)}(x)\, wp\, \sigma_j\,(p)$$

where $\sigma_j(p) = \sum_{n=0}^{p} a_n \, f^h \, (n+1)_{p-n} / (p-n)!$

which was derived by [1].

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